

Numerical quadratures for near-singular and near-hypersingular integrals in boundary element methods

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Abstract

A method of deriving quadrature rules has been developed which gives nodes and weights for a Gaussian-type rule which integrates functions of the form:

$$f(x, y, t) = \frac{a(x, y, t)}{(x-t)^2 + y^2} + \frac{b(x, y, t)}{[(x-t)^2 + y^2]^{1/2}} + c(x, y, t) \log[(x-t)^2 + y^2]^{1/2} + d(x, y, t),$$

without having to explicitly analyze the singularities of $f(x, y, t)$ or separate it into its components. The method extends previous work on a similar technique for the evaluation of Cauchy principal value or Hadamard finite part integrals, in the case when $y \equiv 0$. The method is tested by evaluating standard reference integrals and its error is found to be comparable to machine precision in the best case.

1 Introduction

An important part of the application of the boundary element method (BEM) to physical problems is the calculation of the field, from the solution on the boundary. This requires the evaluation of integrals which contain ‘almost-singular’ integrands which are not properly handled by standard Gaussian quadratures. For example, if we consider a two-dimensional potential problem, such as the Laplace or Helmholtz equations:

$$\nabla^2 \phi = 0, \tag{1a}$$

$$\nabla^2 \phi + k^2 \phi = 0, \tag{1b}$$

where k is the wavenumber, the potential ϕ at some point in the field will be:

$$\phi(\mathbf{x}) = \int_{\Gamma} \frac{\partial \phi}{\partial n} G(\mathbf{x}, \mathbf{x}_1) - \frac{\partial G(\mathbf{x}, \mathbf{x}_1)}{\partial n} \phi \, d\Gamma, \tag{2}$$

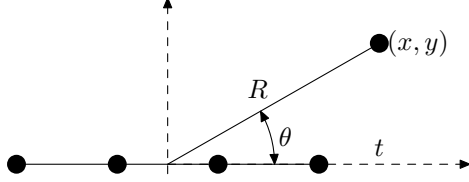


Figure 1: Geometry of boundary element: collocation points indicated by filled circles, field point position $x = R \cos \theta$, $y = R \sin \theta$.

where $G(\mathbf{x}, \mathbf{x}_1)$ is the Green's function for the problem, Γ is the boundary of the domain with normal n and variables of integration are indicated by subscript 1. In the case of the Helmholtz equation, the singular behaviour of the Green's function will be related to the Green's function of the corresponding Laplace equation [8, for example]. The Green's functions for the Laplace equation will have a logarithmic singularity in the planar and axisymmetric case (where the Green's function is proportional to an elliptic integral) and also in the case of an asymmetric problem in an axisymmetric domain [3, 5].

The Green's function for the two-dimensional Laplace equation is $G = \log |\mathbf{x} - \mathbf{x}_1|$. If the boundary integral problem has been solved using the BEM, the evaluation of the potential in the field gives rise, on each element, to an integral of the form:

$$I^{(0)}(x, y) = \int_{-1}^1 (\log |(x-t)^2 + y^2| + f(x, y, t))L(t) dt, \quad (3)$$

where t is the local coordinate on the element, $L(t)$ is a shape function, typically a polynomial, and x and y are the field point coordinates in the local coordinate system, figure 1. This integral is non-singular but suffers from an 'offstage singularity' [1] if the field point is near the element, so that the argument of the logarithm becomes small.

If it is required to determine the first or second derivatives of the potential, for example, to determine a velocity or velocity gradient in a fluid-dynamical problem, integrals of the form:

$$I^{(1)}(x, y) = \int_{-1}^1 \frac{L(t)}{[(x-t)^2 + y^2]^{1/2}} + L(t)g(x, y, t) dt, \quad (4a)$$

$$I^{(2)}(x, y) = \int_{-1}^1 \frac{L(t)}{(x-t)^2 + y^2} + L(t)h(x, y, t) dt, \quad (4b)$$

arise. By analogy with the case when $y \equiv 0$, we refer to these integrals as 'near-singular' and 'near-hypersingular' respectively. When $y \equiv 0$, the field point lies on the boundary and the integrals must be treated as Cauchy principal values [12, page 37] or as a Hadamard finite-part [12, page 31]. In this case, a number of procedures exist for the accurate evaluation of the integrals [4, 13, 14, for example]

A recently-developed method [6, 11] develops quadrature rules which can be used as 'plug-in' replacements for standard Gaussian quadratures in cases where

the integrand contains singularities of the form of equations (3) and (4). An important feature of these rules is that they require no analysis of the integrand before use. When the potential is differentiated, yielding equation (4a), the function $g(x, y, t)$ will usually contain logarithmic singularities which must be properly accounted for, as well as the leading singularity. In many problems, the Green's function will be complicated and the explicit identification of the singularities will be time-consuming. It is preferable to have a quadrature rule which correctly integrates all of the singularities present without requiring a detailed analysis. The previously published method [6, 11], generates such rules for singular and hypersingular integrals. This paper extends this method to 'near-singular' and 'near-hypersingular' integrals, giving a simple, easily implemented technique producing quadrature rules which are direct replacements for Gaussian quadrature in BEM codes.

2 Quadrature rules for near-singular integrals

Before developing the quadrature rules needed for near-singular integrals, it is worth examining the reason for the breakdown of standard Gaussian quadrature. This will also give an indication of when specialized rules should not be used and standard quadratures are better. Consider the integral:

$$\begin{aligned} I &= \int_{-1}^1 \frac{1}{[(x-t)^2 + y^2]^{1/2}} dt, \\ &= \frac{1}{R} \int_{-1}^1 \frac{1}{((t/R)^2 - 2t \cos \theta / R + 1)^{1/2}} dt, \end{aligned} \quad (5)$$

where $R^2 = x^2 + y^2$ and $\theta = \tan^{-1} y/x$. This can be expanded using the generating function for Legendre polynomials [9, 8.921]:

$$\begin{aligned} \frac{1}{(1 - 2zt + t^2)^{1/2}} &= \sum_{k=0}^{\infty} t^k P_k(z), \quad |t| < \min |z \pm (z^2 - 1)^{1/2}|, \\ &\sum_{k=0}^{\infty} t^{-(k+1)} P_k(z), \quad |t| > \max |z \pm (z^2 - 1)^{1/2}|, \end{aligned} \quad (6)$$

to yield:

$$\frac{1}{[(x-t)^2 + y^2]^{1/2}} = \sum_{k=0}^{\infty} \frac{t^k}{R^{k+1}} P_k(\cos \theta), \quad t < R \quad (8a)$$

$$= \sum_{k=0}^{\infty} \frac{R^k}{t^{k+1}} P_k(\cos \theta), \quad t > R. \quad (8b)$$

An N -point Gaussian quadrature rule integrates exactly polynomials up to order $2N - 1$. When $R \gg 1$, the integrand is well approximated by equation (8a) and the estimate of I returned by Gauss-Legendre quadrature:

$$I \approx 2 \sum_{k=0}^N \frac{1}{(2k+1)R^{2k+1}} P_{2k}(\cos \theta), \quad (9)$$

will be accurate. If, however, $R < 1$, part of the integrand will be given by the inverse power series of equation (8b) which cannot be correctly integrated

by the standard Gaussian quadrature. Likewise, even if $R > 1$ but is not large enough to make the terms of equation (8a) decay fast enough, there will be a large error in the estimate of I . Looking ahead to the results presented in §3, it is expected that standard Gauss-Legendre quadrature will give good results for large R and/or in cases where $P_n(\cos \theta)$ is small. Otherwise, a specialized rule will be necessary.

2.1 Evaluation of quadrature rules

The algorithm to be developed gives an N -point rule which integrates a function of the form:

$$f(x, y, t) = \frac{a(x, y, t)}{(x-t)^2 + y^2} + \frac{b(x, y, t)}{[(x-t)^2 + y^2]^{1/2}} + c(x, y, t) \log[(x-t)^2 + y^2]^{1/2} + d(x, y, t), \quad (10)$$

where a , b , c and d are taken to be polynomials of order up to M and the integral

$$I(x, y) = \int_{-1}^1 f(x, y, t) dt \approx \sum_{i=0}^N w_i f(x, y, t_i), \quad (11)$$

where t_i are the quadrature points of an N -point Gauss-Legendre quadrature and the weights w_i are to be determined. A previously-developed algorithm [6, 11] for the computation of quadrature rules gives a method for the evaluation of w_i when $y \equiv 0$. The approach is conceptually simple—the weights are found as the solution to the system of equations:

$$\sum_j [\psi_{ij}] w_j = m_i, \quad i = 1, \dots, 4M, \quad (12)$$

where ψ_{ij} are the weighted Legendre polynomials at the quadrature points t_j :

$$\psi_{ij} = \begin{cases} P_{i-1}(t_j) & 1 \leq i \leq M, \\ P_{i-M-1}(t_j) \log[(x-t_j)^2 + y^2]^{1/2} & M+1 \leq i \leq 2M, \\ P_{i-2M-1}(t_j) / [(x-t_j)^2 + y^2]^{1/2} & 2M+1 \leq i \leq 3M, \\ P_{i-3M-1}(t_j) [(x-t_j)^2 + y^2] & 3M+1 \leq i \leq 4M. \end{cases} \quad (13)$$

and the moments m_i are the integrals of ψ_i

$$m_i = \begin{cases} \int_{-1}^1 P_{i-1}(t) dt & 1 \leq i \leq M, \\ \int_{-1}^1 P_{i-M-1}(t) \log[(x-t)^2 + y^2]^{1/2} dt & M+1 \leq i \leq 2M, \\ \int_{-1}^1 P_{i-2M-1}(t) / [(x-t)^2 + y^2]^{1/2} dt & 2M+1 \leq i \leq 3M, \\ \int_{-1}^1 P_{i-3M-1}(t) [(x-t)^2 + y^2] dt & 3M+1 \leq i \leq 4M. \end{cases} \quad (14)$$

The system of equations is solved using the appropriate LAPACK solver [2], in the least squares sense when $N > 4M$ and in the minimum norm sense when $N < 4M$. The only issue which must be clarified is the evaluation of the moments, m_i . In the case when $y \equiv 0$, the integrals are true Cauchy principal

values or Hadamard finite parts and there exist formulae for their evaluation in terms of associated Legendre functions [10] or simple finite part integrals combined with exact Gaussian quadratures [6]. In this case, however, such simple formulae are not available and a different approach is required.

2.2 Integration of weighted Legendre polynomials

The method outlined in §2.1 for the evaluation of the quadrature weights w_i requires the evaluation of the moments m_i where

$$m_n = \int_{-1}^1 u(t) P_n(t) dt, \quad (15)$$

where weighting function $u(t)$ will be one of $\log[(x-t)^2 + y^2]^{1/2}$ or $[(x-t)^2 + y^2]^{-n/2}$, $n = 1$ or 2 . A general method can be applied to finding the integrals of weighted Legendre polynomials, using basic functional relations and simple formulae for the integrals of elementary functions, readily found in standard references [9].

Assuming that integrals of the form

$$J_n = \int_{-1}^1 t^n u(t) dt, \quad (16)$$

can be evaluated (the formulae required for this paper are given in the appendix), the expansion of powers of t in terms of Legendre polynomials [9, 8.922.1]:

$$t^{2n} = \frac{1}{2n+1} P_0(t) + \sum_{k=1}^n (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(t) \quad (17a)$$

$$t^{2n+1} = \frac{3}{2n+3} P_1(t) + \sum_{k=1}^n (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(t) \quad (17b)$$

can be used to show that:

$$J_{2n} = \frac{1}{2n+1} \int_{-1}^1 P_0(t) u(t) dt + \sum_{k=1}^n (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} \int_{-1}^1 P_{2k}(t) u(t) dt, \quad (18a)$$

$$J_{2n+1} = \frac{3}{2n+3} \int_{-1}^1 P_1(t) u(t) dt + \sum_{k=1}^n (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} \int_{-1}^1 P_{2k+1}(t) u(t) dt. \quad (18b)$$

Then, given the integrals J_i , $i = 0, 1, \dots, N$, the corresponding integrals of the weighted Legendre polynomials, m_i , can be evaluated via:

$$C_i(2n+1)m_{2n} = J_{2n} - J_0/(2n+1) - \sum_{j=1}^n (4j+1)C_j m_{2j}, \quad (19)$$

$$m_0 = J_0, \quad C_j = \frac{2n-2j}{2n+2j+3} C_{j-1}, \quad C_0 = \frac{2n}{2n+3}$$

and

$$D_i(2n+3)m_{2n+1} = J_{2n+1} - 3J_1/(2n+3) - \sum_{j=1}^n (4j+3)D_j m_{2j+1}, \quad (20)$$

$$m_1 = J_1, \quad D_j = \frac{2n-2j}{2n+2j+5} D_{j-1}, \quad D_0 = \frac{2n}{2n+3}.$$

The evaluation can be carried out in place, with values of m_i overwriting J_i . In double precision, the procedure gives accurate results for $n \lesssim 32$ before overflow errors cause problems. For BEM applications, where the shape functions are typically of order no greater than 3, this causes no special difficulties, but would limit use of the method in other areas. Obviously, for any application other than straight elements with $G = \log |\mathbf{x} - \mathbf{x}_1|$, a rule with $M > 3$ will be needed, but it is unlikely that M will need to be greater than 32.

2.3 Summary of method

To summarize, the procedure for computing a quadrature rule with N points which can integrate functions of the form of equation (10) where a , b , c and d are polynomials of order up to M is as follows:

1. find the quadrature points t_i for an N -point Gaussian quadrature, using, for example, the method of Davis and Rabinowitz [7];
2. evaluate the weighted Legendre polynomials ψ_{ij} at each of the quadrature points;
3. compute the moments m_i using the method of §2.2;
4. solve $[\psi_{ij}]w_j = m_i$ using an appropriate solver.

The integral of $f(x, y, t)$ is then approximated by:

$$\int_{-1}^1 f(x, y, t) dt \approx \sum_{i=0}^N w_i f(x, y, t_i). \quad (21)$$

3 Numerical tests

The quadrature method developed in §2 is tested by applying it to the evaluation of a reference integral. Before carrying out these tests, it is of some interest to examine the behaviour of the quadrature weights w_i with respect to the field point position. Figure 2 shows the root mean square difference δ between the rule of this paper with $N = 64$, $M = 16$ and a standard 64-point Gaussian quadrature, with

$$\delta = \left[\frac{1}{N} \sum_{i=1}^N (w_i - v_i)^2 \right]^{1/2}, \quad (22)$$

where v_i are the weights of the standard rule.

The difference between the rules is shown as a function of R and θ . It is clear that close to the element (R and/or θ small), the weights of the new quadrature are very large, as they have to cancel the large value of the integrand close to the near-singularity. Further from the element, however, the integrand is better approximated by a polynomial and the weights come closer to those of the standard rule.

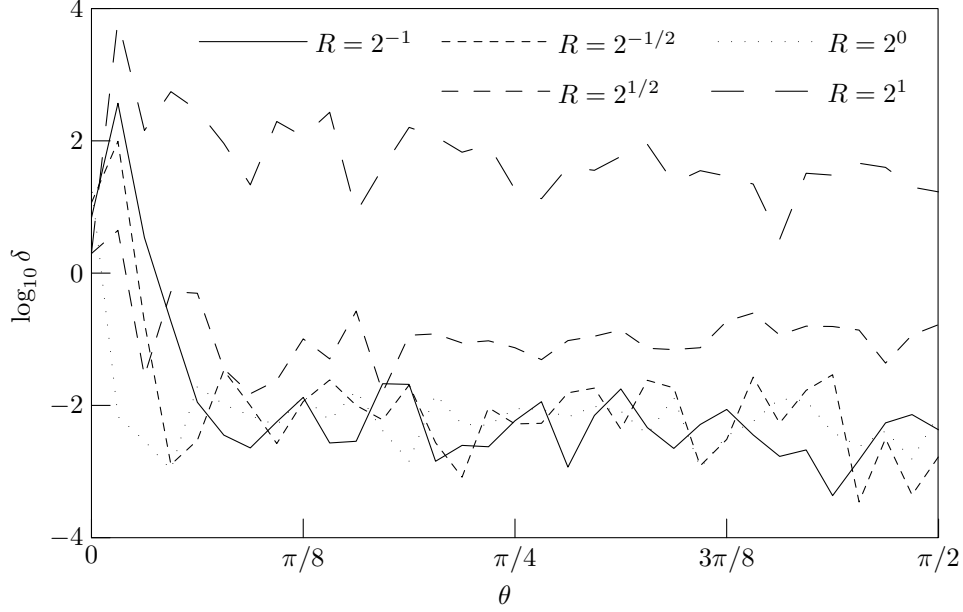


Figure 2: Deviation of quadrature weights from corresponding Gauss-Legendre rule: $N = 64$, $M = 16$.

3.1 Accuracy

The test of accuracy is to examine how well the new quadrature rule evaluates a reference integral. Table 1 shows the error in computing:

$$I = \int_{-1}^1 \frac{t^n}{(x-t)^2 + y^2} dt, \quad (23)$$

for three values of R over the range $\pi/64 \leq \theta \leq 31\pi/64$. The case of $\theta = \pi/2$ was ignored because for odd n , $I \equiv 0$ which would not allow for a meaningful estimate of relative error which is defined:

$$\epsilon = \left[\frac{1}{N} \sum_{i=0}^N \frac{(K(\theta_i) - I(\theta_i))^2}{I(\theta_i)^2} \right]^{1/2}, \quad (24)$$

where K is the value of I estimated by numerical quadrature.

Tables 1 and 2 show the error ϵ_1 , incurred using the method of this paper, compared to ϵ_2 , the error using standard Gauss-Legendre quadrature, using low and high order rules. The calculation is carried out at three values of R to examine the change in error with distance from the element and for various values of n . Table 1 shows the error for $n = 0, \dots, 3$, the important range for boundary element calculations, and a rule with $N = 16$ and $M = 4$. From the first two columns of errors, it is clear that the modified rule is far superior to

Table 1: Root-mean-square error in reference integral computed with modified (ϵ_1) and standard (ϵ_2) quadratures at three values of R , $N = 16$, $M = 4$.

R	2^{-1}		2^0		2^1	
n	ϵ_1	ϵ_2	ϵ_1	ϵ_2	ϵ_1	ϵ_2
0	1.6×10^{-12}	5.9×10^0	3.6×10^{-11}	1.4×10^{-2}	2.1×10^{-16}	4.5×10^{-16}
1	2.8×10^{-13}	3.3×10^0	1.3×10^{-10}	1.3×10^{-2}	1.3×10^{-16}	7.3×10^{-16}
2	6.0×10^{-14}	1.9×10^0	1.0×10^{-10}	1.3×10^{-2}	2.6×10^{-16}	1.5×10^{-15}
3	1.9×10^{-13}	1.0×10^0	9.9×10^{-11}	1.2×10^{-2}	6.3×10^{-16}	3.0×10^{-15}

Table 2: Root-mean-square error in reference integral computed with modified (ϵ_1) and standard (ϵ_2) quadratures at three values of R , $N = 64$, $M = 16$.

R	2^{-1}		2^0		2^1	
n	ϵ_1	ϵ_2	ϵ_1	ϵ_2	ϵ_1	ϵ_2
0	1.9×10^{-10}	1.1×10^{-01}	6.6×10^{-15}	5.6×10^{-12}	1.9×10^{-12}	3.7×10^{-16}
3	3.4×10^{-10}	8.4×10^{-03}	5.4×10^{-15}	4.2×10^{-12}	6.5×10^{-13}	3.0×10^{-15}
6	4.0×10^{-10}	3.8×10^{-03}	7.1×10^{-15}	2.6×10^{-12}	2.3×10^{-12}	2.5×10^{-14}
9	4.0×10^{-10}	8.0×10^{-04}	8.5×10^{-15}	9.9×10^{-13}	1.0×10^{-12}	2.1×10^{-13}
12	4.0×10^{-10}	1.4×10^{-04}	8.4×10^{-15}	6.6×10^{-13}	2.8×10^{-12}	1.7×10^{-12}
15	3.8×10^{-10}	2.2×10^{-05}	8.3×10^{-15}	2.3×10^{-12}	5.2×10^{-12}	1.3×10^{-11}

the standard quadrature: its error is about twelve orders of magnitude lower than that of the Gaussian quadrature. Similarly, for $R = 1$, the mean error is much smaller, being no worse than about 10^{-10} , rather than 10^{-2} . Once the field point is far from the element, however, at $R = 2$, both rules have similar accuracy.

Table 2 shows similar results for a rule with $N = 64$ and $M = 16$, compared to data for a standard Gauss-Legendre rule with $N = 64$. Integrals of order up to $n = 15$ have been computed and, as before, at small R , the error behaviour of the new rule is orders of magnitude better than that of the standard technique. At larger R , however, the advantage is not so clear cut: at $R = 1$, the error in the standard rule is around 10^{-12} , still larger than that from the method of this paper, but probably acceptable in many applications. When $R = 2$, the Gauss-Legendre rule gives results comparable to those of the new technique, although it performs better on low order polynomials. As might be expected, when the distance from the element is large enough, a high order Gauss-Legendre rule can capture enough of the behaviour of the integrand to accurately compute the integral.

To examine the error behaviour in more detail, figures 3–5 show the relative

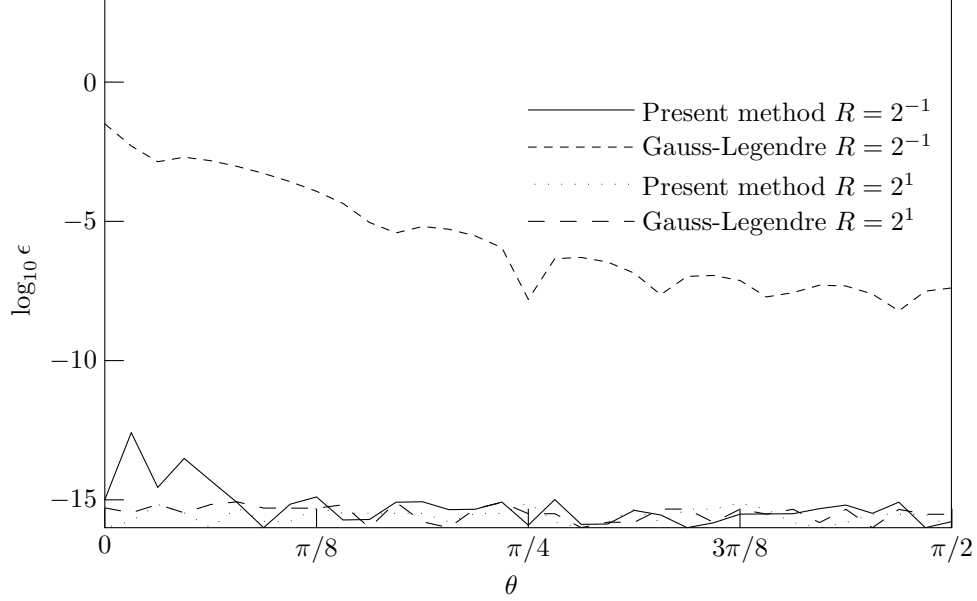


Figure 3: Relative error in logarithmically singular integral, $N = 16$, $M = 4$.

error in computing:

$$I_0^{(\log)}(x, y) = \int_{-1}^1 \log[(x-t)^2 + y^2]^{1/2}, \quad (25)$$

$$I_0^{(1)}(x, y) = \int_{-1}^1 [(x-t)^2 + y^2]^{-1/2}, \quad (26)$$

$$I_0^{(2)}(x, y) = \int_{-1}^1 [(x-t)^2 + y^2]^{-1}, \quad (27)$$

with error ϵ defined as:

$$\epsilon = \left| \frac{I_0 - J_0}{I_0} \right|, \quad (28)$$

where I_0 is one of $I_0^{(\log)}(x, y)$, $I_0^{(1)}(x, y)$, $I_0^{(2)}(x, y)$ and J_0 is the corresponding estimate using the quadrature rule. In each case, $N = 16$ and $M = 4$ and a sixteen point Gaussian quadrature was also applied for comparison. In each case, for large distances from the element, $R = 2$, both quadrature techniques are accurate, with errors of the order of machine precision. When $R = 2^{-1}$, however, the error incurred using Gaussian quadrature is unacceptably large, while the modified rule gives very accurate answers, again of the order of machine precision. As might be expected, the error from the Gaussian quadrature is smaller as $\theta \rightarrow \pi/2$, due to the greater distance from the element, but it is never better than about 10^{-6} , insufficient accuracy for most applications.

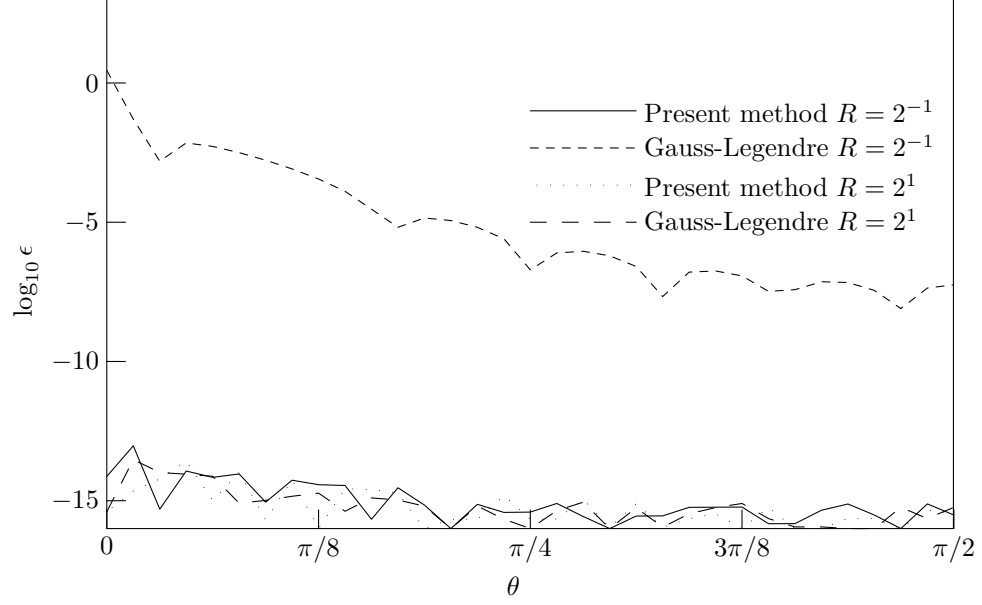


Figure 4: Relative error in near-singular integral, $N = 16$, $M = 4$.

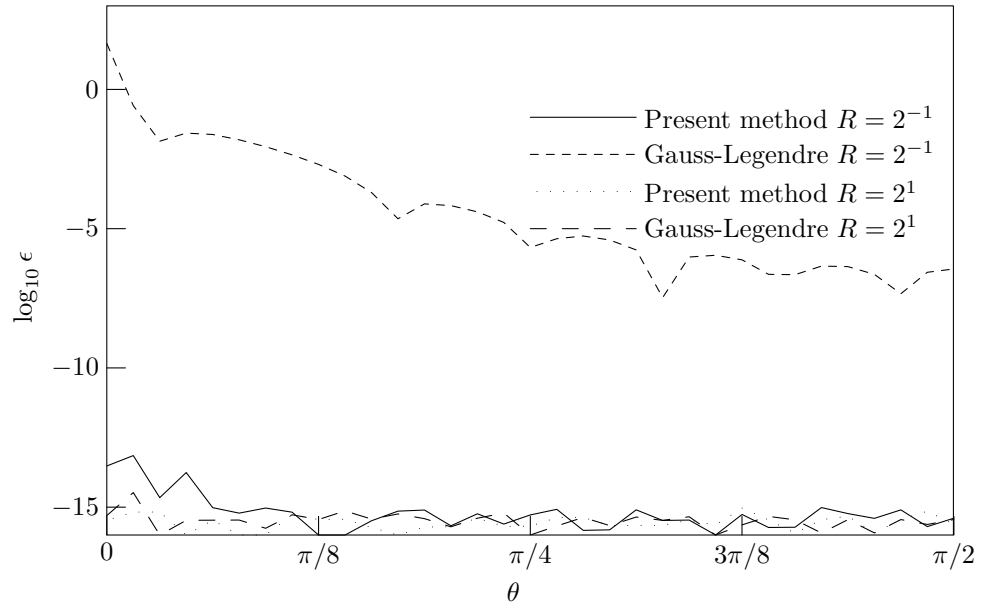


Figure 5: Relative error in near-hypersingular integral, $N = 16$, $M = 4$.

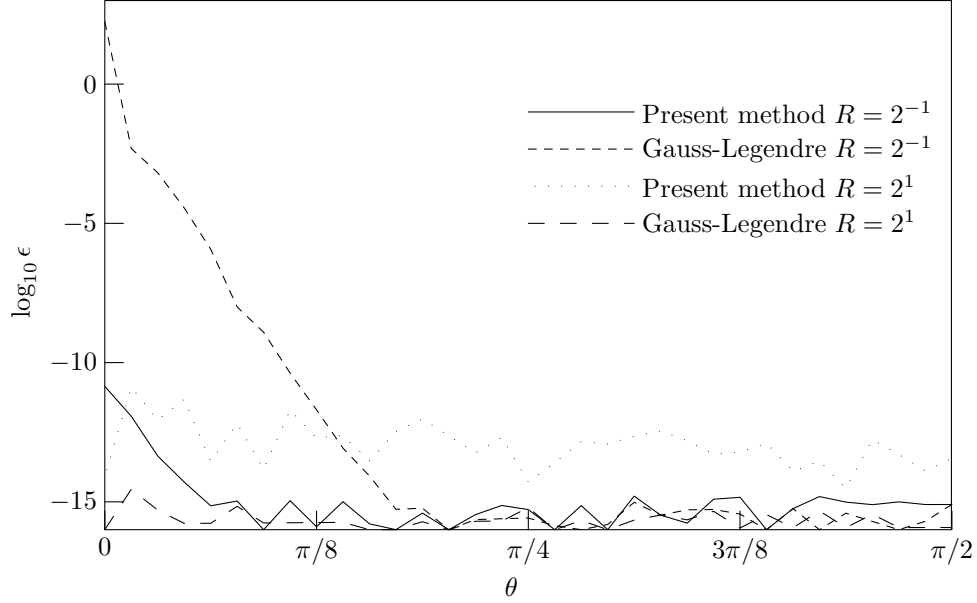


Figure 6: Relative error in near-hypersingular integral, $N = 64$, $M = 16$.

Finally, figure 6 illustrates an interesting point about the error behaviour of the quadrature rule as the number of quadrature points is increased. It shows the error in the near-hypersingular integral $I_0^{(2)}(x, y)$ evaluated using a rule with $N = 64$ and $M = 16$, with the error from a 64-point Gaussian quadrature shown for comparison. The first point is that the Gaussian quadrature is able to cope with the singularity for $R = 2^{-1}$ when $\theta \gtrsim \pi/8$, because it can integrate polynomials of high enough order to be able to correctly handle the expansion of the integrand in Legendre polynomials. Also, as in the previous cases, it can correctly evaluate the integral for $R = 2$.

The modified rule, however, has slightly worse error behaviour than for $N = 4$, with the maximum error being higher for $R = 2^{-1}$ and the error at $R = 2$ being greater across the full range of θ . The error is still small, being less than 10^{-10} , but the reason for the increase is unclear. It appears to be due to an ambiguity in expressing the integrand in terms of Legendre polynomials: the near-hypersingular part of the integrand $f(t)/[(x-t)^2 + y^2]$ is a ratio of two polynomials which can be written as a sum of a proper elementary function and a polynomial. This polynomial term is then represented twice in the quadrature rule, being handled by the unweighted Legendre polynomials, m_i , $1 \leq i \leq M$ in equation (14) and by the weighted polynomials. An interesting question for future developments of the technique will be how best to choose the elementary functions for the quadrature rule to give optimal accuracy for a given N .

4 Conclusions

A method of deriving quadrature rules for the evaluation of the ‘near-singular’ integrals which arise in the boundary element method has been derived. The performance of the technique has been assessed by evaluation of reference integrals and it has been found that it outperforms standard Gaussian quadrature rules with the same number of nodes for field points close to the element. The error in the integral increases slightly with the number of points in the rule, a point which is to be investigated in future work.

A Integrals of weighted polynomials

To evaluate the required integrals of Legendre polynomials using the procedure of §2.2, we require a method of evaluating the integrals:

$$I_n^{(2)}(x, y) = \int_{-1}^1 \frac{t^n}{(x-t)^2 + y^2} dt, \quad (29a)$$

$$I_n^{(1)}(x, y) = \int_{-1}^1 \frac{t^n}{[(x-t)^2 + y^2]^{1/2}} dt, \quad (29b)$$

$$I_n^{(\log)}(x, y) = \int_{-1}^1 t^n \log[(x-t)^2 + y^2]^{1/2} dt, \quad (29c)$$

Use of standard formulae [9, 2.171, 2.263, 2.728.1] yields:

$$I_n^{(2)}(x, y) = \frac{1+(-1)^n}{n-1} + 2xI_{n-1}^{(2)}(x, y) - R^2I_{n-2}^{(2)}(x, y), \quad (30a)$$

$$I_n^{(1)}(x, y) = \frac{R_+ - R_-}{n} + \frac{2n-1}{n}xI_{n-1}^{(1)}(x, y) - \frac{n-1}{n}R^2I_{n-2}^{(1)}(x, y), \quad (30b)$$

$$I_n^{(\log)}(x, y) = \frac{\log R_+ + (-1)^n \log R_-}{2(n+1)} - \frac{1}{n+1}I_n^{(2)}(x, y) + \frac{R \cos \theta}{n+1}I_n^{(2)}(x, y), \quad (30c)$$

where the recursions are seeded with:

$$I_0^{(2)}(x, y) = \frac{1}{y} \left(\tan^{-1} \frac{1-x}{y} - \tan^{-1} \frac{1+x}{y} \right), \quad I_1^{(2)}(x, y) = \log \frac{R_+}{R_-} + xI_0^{(2)}(x, y),$$

$$I_0^{(1)}(x, y) = \log \frac{R_+ + 1 - x}{R_- - 1 - x}, \quad I_1^{(1)}(x, y) = R + xI_0^{(1)}(x, y),$$

and

$$R = (x^2 + y^2)^{1/2}, \quad R_{\pm} = [(x \mp 1)^2 + y^2]^{1/2}, \quad \theta = \tan^{-1} y/x.$$

References

- [1] Forman S. Acton. *Numerical methods that work*. Mathematical Association of America, 1990.
- [2] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen. *LAPACK Users' Guide*. Society for Industrial and Applied Mathematics, Philadelphia, PA, third edition, 1999.

- [3] Jonas Björkberg and Gerhard Kristensson. Electromagnetic scattering by a perfectly conducting elliptic disk. *Canadian Journal of Physics*, 65:723–734, 1987.
- [4] Mauricio Pazini Brandão. Improper integrals in theoretical aerodynamics: The problem revisited. *AIAA Journal*, 25(9):1258–1260, September 1987.
- [5] Michael Carley. Scattering by quasi-symmetric pipes. *Journal of the Acoustical Society of America*, 119(2):817–823, February 2006.
- [6] Michael Carley. Numerical quadratures for singular and hypersingular integrals in boundary element methods. *SIAM Journal on Scientific Computing*, 29(3):1207–1216, 2007.
- [7] Philip J. Davis and Philip Rabinowitz. *Methods of numerical integration*. Academic, New York, 1975.
- [8] Trevor W. Dawson. On the singularity of the axially symmetric Helmholtz Green’s function, with application to BEM. *Applied Mathematical Modelling*, 19:590–599, October 1995.
- [9] I. Gradshteyn and I. M. Ryzhik. *Table of integrals, series and products*. Academic, London, 5th edition, 1980.
- [10] A. C. Kaya and F. Erdogan. On the solution of integral equations with strongly singular kernels. *Quarterly of Applied Mathematics*, XLV(1):105–122, April 1987.
- [11] P. Kolm and V. Rokhlin. Numerical quadratures for singular and hypersingular integrals. *Computers and Mathematics with Applications*, 41:327–352, 2001.
- [12] M. J. Lighthill. *An introduction to Fourier analysis and generalised functions*. Cambridge University Press, Cambridge, 1958.
- [13] Giovanni Monegato. On the weights of certain quadratures for the numerical evaluation of Cauchy principal value integrals and their derivatives. *Numerische Mathematik*, 50:273–281, 1987.
- [14] Giovanni Monegato. Numerical evaluation of hypersingular integrals. *Journal of Computational and Applied Mathematics*, 50:9–31, 1994.